AP CALCULUS AB | Homework 4.3 | Name:
DR. PAUL L. BAILEY | Monday, January 6, 2020

Write your homework *neatly, in pencil,* on blank white $8\frac{1}{2} \times 11$ printer paper. Always *write the problem,* or at least enough of it so that your work is readable. When appropriate, *write in sentences.*

The main theory of section 4.3 is summarized below.

Definition 1. Let f be a function defined on an interval I.
We say that f is *increasing on I* if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.
We say that f is *decreasing on I* if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.

The following is another corollary to the Mean Value Theorem.

Corollary 3. *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*
If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

Theorem 1. (First Derivative Test)
Suppose that f is differentiable in on interval containing c, and that $f'(c) = 0$.

- *If f' changes sign from positive to negative at c, then f has a local maximum at c.*
- *If f' changes sign from negative to positive at c, then f has a local minimum at c.*
- *If f' does not change sign at c, then f does not have a local extremum at c.*

Problem 1 (Thomas §4.3 # 1). Let

$$f'(x) = x(x - 1).$$

What are the critical points of f? On what intervals is f increasing or decreasing? At points, if any, does f attain local maximum or minimum values?

Problem 2 (Thomas §4.3 # 4). Let

$$f'(x) = (x - 1)^2(x + 2)^2.$$

What are the critical points of f? On what intervals is f increasing or decreasing? At points, if any, does f attain local maximum or minimum values?

Problem 3 (Thomas §4.3 # 12). Let

$$h(x) = 2x^3 - 18x.$$

Find the intervals on which the function is increasing or decreasing. Identify the points in the domain where h has a local extremum, and find the local extreme value. Which, if any, of the extreme values are absolute? Sketch the graph of h.

Problem 4 (Thomas §4.3 # 21). Let

$$g(x) = x\sqrt{8 - x^2}.$$

Find the intervals on which the function is increasing or decreasing. Identify the points in the domain where g has a local extremum, and find the local extreme value. Which, if any, of the extreme values are absolute?

Problem 5 (Thomas §4.3 # 48). Find the intervals on which $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is increasing and decreasing. Describe the reasoning behind your answer.

Here is another definition we need.

Definition 2. Let f be continuous on $[a, b]$. The *average rate of change* of f on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

This is the slope of the line from $(a, f(a))$ to $(b, f(b))$.

Problem 6. Let

$$f(x) = \frac{x^2 + 1}{x}.$$

(a) Let $g(b)$ denote the average rate of change of f on $[1, b]$. Write $g(b)$ as a function of b. Simplify.

(b) Find $\lim_{b \to \infty} g(b)$.

Problem 7. Let

$$f(x) = x^3 - 6x^2 + 9x.$$

(a) Find the zeros of f.

(b) Find the zeros of f'.

(c) Find the points on the graph of f where f has local extreme values.

(d) Use this information to sketch the graph of f.

Problem 8. Let

$$f(x) = \frac{e^x + e^{-x}}{2}.$$

Show that $f''(x) = f(x)$ for all $x \in \mathbb{R}$.

Problem 9. Let a be a positive real number and let $y = a^x$. Then $\ln y = \ln(a^x) = x \ln a$.

$$a^x = y = e^{\ln y} = e^{x \ln a};$$

that is,

$$a^x = e^{x \ln a}.$$

Use this to compute $\dfrac{d}{dx} a^x$.

Problem 10. The Three Great VT's are the Intermediate Value Theorem (IVT), the Extreme Value Theorem (EVT), and the Mean Value Theorem (MVT).

(a) Precisely write the hypothesis and conclusion of each of these theorems.

(b) Match the following physical situation with the given theorem.

- A motorist passed a policeman at 10:15 AM and passed his partner 10 miles down the road at 10:25. He was arrested for speeding at 60 miles per hour.
- The chicken had to cross the road to get to the other side.
- A man was born weighing 7 pounds, became a rich glutton, but died of starvation; yet, at some point in his life, he attained a maximum weight.

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#1) $f'(x) = x(x-1)$ f': $+$ | $-$ | $+$ at 0, 1

cp: $0, 1$

incr on $(-\infty, 0) \cup (1, \infty)$

decr on $(0, 1)$

local max at 0

local min at 1

#2) $f'(x) = (x-1)^2(x+2)^2$ $+$ | $+$ | $+$ at -2, 1

cp: $-2, 1$

incr on $\mathbb{R}$ no local extrema

#3) let $h(x) = 2x^3 - 18x = 2x(x^2-9) = 2x(x-3)(x+3)$

zeros: $0, \pm 3$

$h'(x) = 6x^2 - 18$

$= 6(x^2 - 3)$

cp: $\pm\sqrt{3}$. h': $+$ | $-$ | $+$ at $-\sqrt{3}$, $\sqrt{3}$

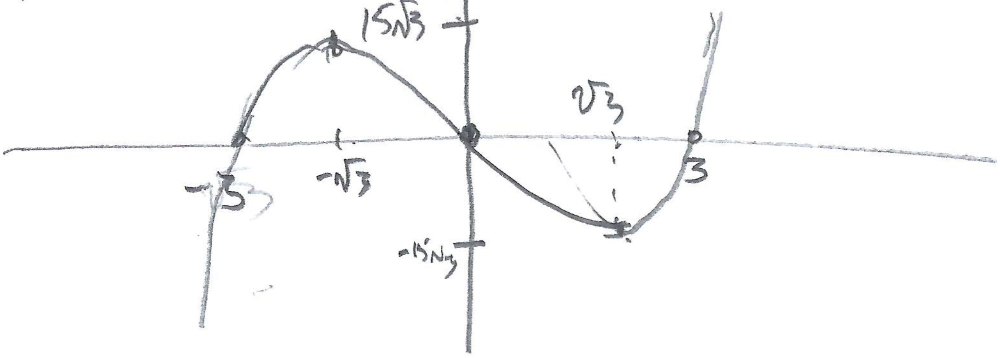
incr on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

decr on $(-\sqrt{3}, \sqrt{3})$

local max at $x = -\sqrt{3}$ $h(-\sqrt{3}) = -6\sqrt{3} + 18\sqrt{3} = 12\sqrt{3}$

$h(\sqrt{3}) = -12\sqrt{3}$

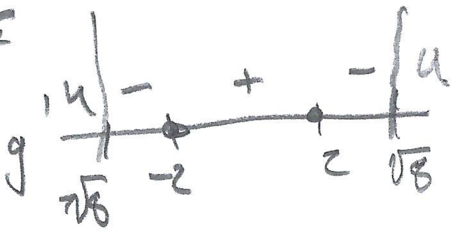
no extremum is absolute



#4 $g(x) = x\sqrt{8-x^2}$

$g'(x) = \sqrt{8-x^2} + x \cdot \frac{1}{2\sqrt{8-x^2}}(-2x)$

$= \frac{8-x^2-x^2}{\sqrt{8-x^2}} = \frac{8-2x^2}{\sqrt{8-x^2}}$



$g' = 0 \to x^2 = 4 \Rightarrow x = \pm 2$

Incr on $(-2, 2)$

Decr on $(-\sqrt{8}, -2) \cup (2, \sqrt{8})$

#5 $f(x) = ax^2 + bx + c$. The vertex is at $\frac{-b}{2a}$

If $a > 0$, decr on $(-\infty, -\frac{b}{2a})$ incr on $(-\frac{b}{2a}, \infty)$

but if $a < 0$, decr on $(-\frac{b}{2a}, \infty)$ incr on $(-\infty, -\frac{b}{2a})$.

#6 $f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$

Avg rate on $[1, b]$ is $\frac{f(b) - f(1)}{b-1} = \frac{b + \frac{1}{b} - 1 - 1}{b-1}$

$= 1 + \frac{1/b - 1}{b-1} = 1 + \frac{1-b}{b(b-1)} = 1 - \frac{1}{b}$ $\quad = \cancel{1 + \frac{1/b - 1}{b-1}}$

So $\lim_{b \to \infty} 1 - \frac{1}{b} = 1$

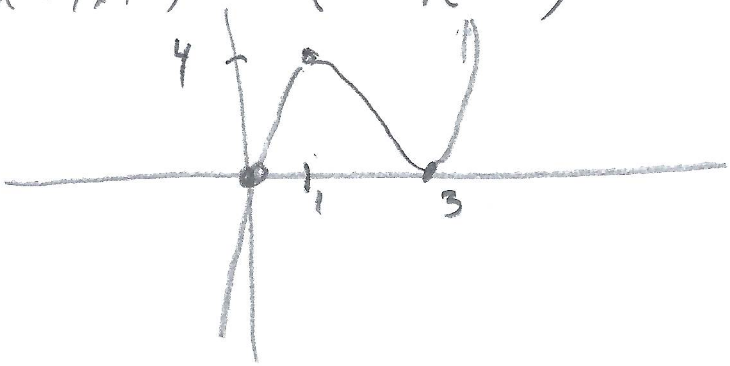
#7 let $f(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

$f(x) = 0 \Rightarrow x = 0, 3$

$f'(x) = 0 \Rightarrow x = 1, 3$

$f(1) = 1 - 6 + 9 = 4$



#8 Let $f(x) = \dfrac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x + e^{-x})$

Then $f'(x) = \frac{1}{2}(e^x - e^{-x})$

So $f''(x) = \frac{1}{2}(e^x - (-e^{-x})) = \dfrac{e^x + e^{-y}}{2} = f(x)$

#9 $\dfrac{d}{dx} a^x = \dfrac{d}{dx} \exp(\log a^x)$

$= \dfrac{d}{dx} \exp(x \log a) = \log a \exp(x \log a)$

$= (\log a) a^x$

#10 EVT Hyp: let f be continuous on $[a,b]$.

Conc: Then f has an absolute min and max on $[a,b]$.

IVT Hyp: Let f be continuous on $[a,b]$ and.

Suppose $f(a) \cdot f(b) < 0$

Conc: Then there exists $c \in (a,b)$ such that $f(c) = 0$.

MVT: Hyp: Let f be continuous on $[a,b]$ and differentiable on (a,b).

Conc: Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

- Motorist speeding: MVT
- Chicken crossing Road: IVT
- Weight: EVT.